J. Fluid Mech. (1966), *vol.* 24, *part* 4, *pp.* 765–767
Printed in Great Britain

Note on edge waves in a stratified fluid

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(Received 25 January 1965 and in revised form 4 October 1965)

The solution given by Gerstner for water waves of finite amplitude, which is valid for a semi-infinite liquid of arbitrary stratification, is reconsidered. By a transformation of co-ordinates, it is shown to represent edge waves propagating along a sloping shore.

1. Gerstner waves in a stratified liquid

Gerstner (1802; see Lamb 1945, p. 421) gave a particular solution for rotational waves of finite amplitude in a semi-infinite liquid of zero viscosity and constant density. It is well known that, in a frame of reference moving with Gerstner waves, the pressure on *any* streamline is a constant, and therefore any streamline can be considered as the trace of the free surface. This fact immediately suggests that Gerstner waves are possible in an inviscid liquid of any stratification in density, provided the depth is infinite as in Gerstner's case, and indeed the correctness of this conclusion has been fully established by Dubreil-Jacotin (1932, p. 819). The analysis occupying the remainder of this section, incorporating this extension of Gerstner's results and in essence following Dubreil-Jacotin's treatment, is presented as a convenient introduction for the new material in § 2.

In terms of Lagrangian co-ordinates a_i $(i = 1, 2, 3)$, which are not necessarily the initial Cartesian co-ordinates, the equations of motion are

$$(\ddot{x}_\alpha - X_\alpha)\frac{\partial x_\alpha}{\partial a_i} + \frac{1}{\rho}\frac{\partial p}{\partial a_i} = 0 \quad (i = 1, 2, 3), \tag{1}$$

in which dots indicate differentiations with respect to the time t, x_i is the ith Cartesian co-ordinate of a fluid particle as it moves about, X_i is the ith component of the body force per unit mass, ρ is the density, and p is the pressure. The repeated indices α imply summation over 1, 2, and 3.

Gerstner considered a two-dimensional flow independent of a_3, and took the direction of increasing x_2 to be vertical, so that

$$X_1 = X_3 = 0, \quad X_2 = -g,$$

where g is the gravitational acceleration. Denoting x_1 and x_2 by x and y, and a_1 and a_2 by a and b, Gerstner showed that, for constant density, the solution consisting of

$$x = a + \frac{1}{k}e^{kb}\sin k(a-ct), \quad y = b - \frac{1}{k}e^{kb}\cos k(a-ct) \tag{2}$$

satisfies (1) and the Lagrangian equation of continuity, and represents waves of finite amplitude propagating in the x-direction with a speed c given by

$$c^2 = g/k, \tag{3}$$

in which k is a wave-number. The pressure is given by

$$P = p/\rho = C_0 - gb + \tfrac{1}{2}c^2 e^{2kb}, \tag{4}$$

in which C_0 is a constant.

For clarity in exposition we shall consider Gerstner's solution for a homogeneous liquid to be a solution satisfying

$$(\ddot{x} - X_\alpha)\frac{\partial x_\alpha}{\partial a_i} + \frac{\partial}{\partial a_i}P = 0 \quad (i = 1, 2), \tag{5}$$

with α ranging over 1 and 2. Now, if ρ is variable, the corresponding two equations in (1) are satisfied if

$$\frac{1}{\rho}\frac{\partial p}{\partial a_i} = \frac{\partial P}{\partial a_i}, \tag{6}$$

or

$$p = f(P), \quad \rho = f'(P), \tag{7}$$

with the accent indicating differentiation with respect to the argument of the arbitrary function f. Since P is a function of b only, this means that the velocity field obtained by Gerstner is dynamically possible even if ρ is not constant, but is a function of b, and that the isopycnic surfaces are isobaric surfaces in Gerstner's flow. The density stratification is entirely arbitrary, and any constant-density surface can be taken to be a free surface.

Equations (6) permit one to write

$$p = \int \rho \, dP. \tag{8}$$

With P given in (4), this becomes

$$p = g\int_0^b \rho(e^{2kb} - 1)\, db = I(b), \tag{9}$$

in which the lower limit of the integral has been chosen so that $p = 0$ for $b = 0$.

2. Edge waves in a stratified fluid

If a co-ordinate system as shown in figure 1 is adopted, in which the direction of x (or x_1) is normal to the plane of the figure, and x_2 and x_3 are designated y and z, the body-force components are

$$X_1 = 0, \quad X_2 = -g\sin\beta = -g_2, \quad X_3 = -g\cos\beta = -g_3. \tag{10}$$

If Gerstner's velocity field is retained, the two equations in (1) for $i = 1$ and 2 are satisfied if

$$\frac{1}{\rho}\frac{\partial p}{\partial a_i} = \frac{\partial}{\partial a_i}(P - g_3 z), \tag{11}$$

with $z = a_3$, and g replaced by g_2 in the expressions for c and P in (3) and (4). The third equation in (1) merely states that p is hydrostatic in the direction of z. But (11) implies that

$$p = f(P - g_3 z), \quad \rho = f'(P - g_3 z). \tag{12}$$

Thus again the isopycnic surfaces are isobaric surfaces, any of which can be taken to be the free surface. The density stratification is again arbitrary. With ρ or p fixed, $P \to \infty$ as $z \to \infty$. Since according to (2) and (4), with g in (4) identified with the present g_2, we have

$$P \sim -g_2 y + \text{const.}, \tag{13}$$

so that $y \to -\infty$ as $P \to \infty$. Thus the surfaces of constant ρ or p are asymptotically normal to the body force $\mathbf{g} = (0, -g_2, -g_3)$, as is to be expected from the vanishing of the acceleration under this limit, that is, as $z' \to \infty$ in figure 1.

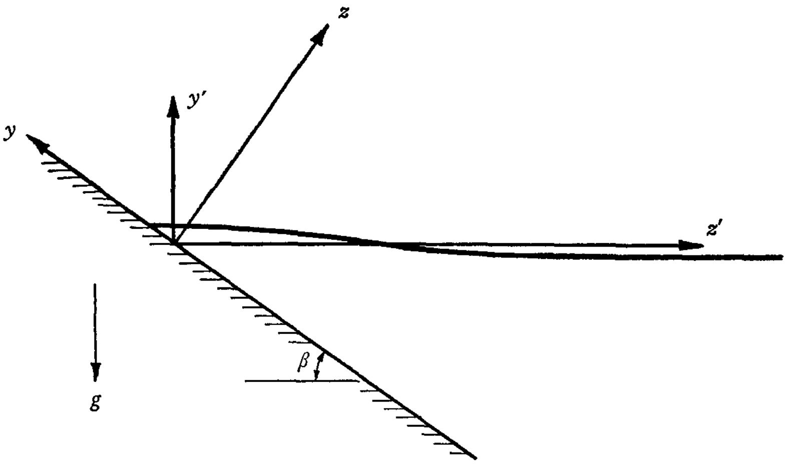

FIGURE 1. Sketch of a cross-section of edge waves of finite amplitude.

That there is no velocity normal to the sloping shore (inclined at an angle β with the horizontal) is obvious, because Gerstner's velocity field is plane, and without any component in the direction of z. It remains to mention that, since g_2 now corresponds to the g in Gerstner's original solution, it follows without further ado that the phase velocity of edge waves is given by

$$c^2 = g \sin\beta / k. \tag{14}$$

This work has been supported by the National Science Foundation and the Army Research Office (Durham). The author wishes to thank the referee of this paper for his helpful criticisms, and Dr T. Brooke Benjamin for directing his attention to the article by Dubreil-Jacotin.

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